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Lower bounds for eigenvalues of the Dirac operator on n-spheres with $so(n)$-symmetry

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Abstract

In this paper we derive estimates for the eigenvalues of the Dirac operator and their multiplicity on manifolds diffeomorphic to S^n with an isometric $SO(n)$-action. Especially we prove a new lower bound for the first eigenvalue and show an example, where this new bound coincides in the limit with the known upper bounds. © 2000 Elsevier Science B.V. All right reserved.

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1. Introduction

The Dirac operator on a Riemannian spin manifold M is a natural first-order elliptic differential operator. For closed M, its spectrum is discrete and real.

There are a few examples, where the spectrum is explicitly computed. For homogeneous spaces the calculation can be reduced to representation theoretic computations, which have been done in several examples. For the spheres they result in the following theorem:

Theorem 1 [21]. *On the sphere S^n of constant sectional curvature* 1 *the Dirac operator has the eigenvalues*

$$\pm\left(\frac{n}{2}+k\right), \quad k \geq 0 \tag{1}$$

with multiplicity $2^{[\frac{n}{2}]}\binom{k+n-1}{k}$.

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More examples, where the spectrum of the Dirac operator is well known are listed in [2], e.g. the flat tori [15], complex projective spaces [12,13,20] and spherical space forms [6].

In general the Dirac operator has a nontrivial kernel (cf. [22], examples cf. [7] and [15]), but for n-dimensional manifolds M^n with positive scalar curvature R one deduces from the Lichnerowicz-formula that the kernel is trivial. In this case Friedrich has proved in [14] that the eigenvalues satisfy

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1} R_{\min} \tag{2}$$

with $R_{\min} := \min\{R(m) | m \in M^n\}$.

For two-dimensional manifolds (M^2, g) of genus zero the Dirac operator has no kernel, because any metric g on S^2 is conformally equivalent to the standard metric and in [5] and [9] further lower bounds for the first eigenvalue have been shown. If (M^2, g) with $M^2 \cong S^2$ has additionally an S^1-symmetry, then in [18] we proved that the first eigenvalue satisfies

$$\lambda \geq \frac{1}{2 f_{\max}}, \tag{3}$$

where $2\pi f_{\max}$ is the maximal volume of S^1-orbits. This result will be generalised in this paper:

Theorem 2. *Let (M^{n+1}, g) be a Riemannian manifold, $M^{n+1} \cong S^{n+1}$ with an isometric $SO(n+1)$-action. Let $f_{\max} \cdot \mathrm{vol}(S^n)$ be the maximal volume of orbits. Then the spectrum of M is symmetric. The multiplicity of an eigenvalue λ with*

$$\frac{(n/2)+k}{f_{\max}} \leq \lambda < \frac{(n/2)+k+1}{f_{\max}}, \quad (k \geq 0)$$

is at most $2^{[\frac{n+1}{2}]}\binom{n+k}{k}$.

Especially the smallest positive eigenvalue satisfies $\lambda \geq n/(2 f_{\max})$ and has multiplicity at most $2^{[(n+1)/2]}$.

There are some well-known upper bounds for the first positive eigenvalue of the Dirac operator, e.g. for compact connected Riemannian spin manifolds of dimension $2m$ and positive sectional curvature K with maximum $K_{\max}$ the inequality

$$\lambda^2 \leq 4^{m-1} m K_{\max}$$

for the first positive eigenvalue holds ([4]). More upper bounds are proved in dimension 2 (cf. [1]) and for manifolds M^n, which are isometrically immersed in $\mathbb{R}^{n+1}$ (e.g. [4,11,17]). We will use the following inequality for the first eigenvalue to show that the new lower bound is sharp in the limit.

Lemma 1 ([8], Corollary 4.2). *Let M be an n-dimensional closed oriented hypersurface isometrically immersed in $\mathbb{R}^{n+1}$. Let M carry the induced spin structure. Let H be the mean curvature of M in $\mathbb{R}^{n+1}$.*

Then the classical Dirac operator D_M of M has at least $2^{[n/2]}$ eigenvalues λ (counted with multiplicities) satisfying

$$\lambda^2 \leq \frac{n^2 \int H^2 \, \mathrm{d}M}{4 \operatorname{vol}(M)}. \tag{4}$$

We will show an example where the new lower bound coincides in the limit with this upper bound (4) even with multiplicity.

In the next section we recall some facts about Dirac operators on hypersurfaces and warped products and apply them to the space of principal orbits. This reduces the Dirac equation on the $SO(n+1)$-manifold S^{n+1} to a system of first-order differential equations. In Section 3, this will be used to prove Theorem 2.

2. The Dirac operator on the bundle of principal orbits

Let (M, g) be an $n+1$-dimensional oriented Riemannian manifold $M \cong S^{n+1}$ with an isometric effective $SO(n+1)$-action. Then the $SO(n+1)$-action has only principal orbits which are spheres S^n and two fixed points ([10], p. 185 and p. 230). We denote with M_0 the dense submanifold of principal orbits.

Let $\gamma : I = (0, L) \to M_0$ be a section of $M_0 \to M_0/SO(n+1)$ orthogonal to the fibres parametrised by arc length and

$$f : I \to \mathbb{R}^+, \quad f(t) = \frac{\operatorname{vol}(SO(n+1) \cdot \gamma(t))}{2\pi^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right)$$

then

$$(M_0, g) = (I \times S^n, \mathrm{d}t^2 + f^2(t)\, \mathrm{d}\phi^2) =: I \times_f S^n,$$

where $\mathrm{d}\phi^2$ denotes the standard metric on S^n. For given f the closure of $I \times_f S^n$ is a C^∞-manifold iff $\dot{f}(0) = -\dot{f}(L) = 1$ and f extends at zero to an odd smooth function and similarly at L.

To calculate the Dirac operator on the space of principal orbits M_0, we use some facts about spinor bundles on hypersurfaces and warped products which have been developed in [3] and similarly in [7, p. 902], with a slight difference in the case n odd. We will follow [7] in this paper. For details concerning the Dirac operator, the references are [16] and [19].

Let X^{n+1} be an $(n+1)$-dimensional spin manifold with spin structure Q_X and $Y^n \subset X^{n+1}$ an n-dimensional closed submanifold with the induced spin structure Q_Y, such that the restriction of the spinor bundle over X to Y is given by

$$\Sigma_X|_Y = Q_Y \times_{\mathrm{Spin}(n)} \Delta_{n+1}$$

where Δ_{n+1} denotes the $(n+1)$-dimensional spinor-representation.

If n is odd Δ_{n+1} admits a splitting $\Delta_{n+1} = \Delta_{n+1}^+ \oplus \Delta_{n+1}^-$ into two irreducible spin representations. The canonical algebra isomorphism $\mathrm{Cl}(\mathbb{R}^n) \to \mathrm{Cl}^0(\mathbb{R}^{n+1})$ of the Clifford algebra of $\mathbb{R}^n$ in the even part of the Clifford algebra of $\mathbb{R}^{n+1}$ induced by $e_i \mapsto e_i \cdot e_{n+1}$ for the standard basis $(e_1, \ldots, e_{n+1})$ of $\mathbb{R}^{n+1}$ makes Δ_{n+1} into two representations of $\mathrm{Cl}(\mathbb{R}^n)$. Therefore the following relation of the spinor bundles Σ_X and Σ_Y over X and Y holds:

$$\Sigma_X|_Y = \Sigma_X^+|_Y \oplus \Sigma_X^-|_Y = \Sigma_Y \oplus \hat{\Sigma}_Y.$$

The induced Clifford multiplication with vectors $v \in TY$ on Σ_Y and $\hat{\Sigma}_Y$ differs in the sign and Clifford multiplication with a normal unit vectorfield ν on Y gives an isomorphism $\Sigma_Y \to \hat{\Sigma}_Y$ which anticommutes with the Clifford multiplication.

For n even, the algebra isomorphism $\mathrm{Cl}(\mathbb{R}^n) \to \mathrm{Cl}^0(\mathbb{R}^{n+1})$ makes Δ_{n+1} into a representation of $\mathrm{Cl}(\mathbb{R}^n)$ which is the usual spinor representation of $\mathrm{Cl}(\mathbb{R}^n)$. Therefore we get

$$\Sigma_X|_Y \cong \Sigma_Y = \Sigma_Y^+ \oplus \Sigma_Y^-.$$

Let ψ be a spinor on X defined on a neighbourhood of Y then the classical Gauss-formula for the Levi-Civita connection gives for the Dirac operator

$$-\nu \cdot D^X\psi = D^Y\psi - \frac{n}{2}H\psi + \nabla_\nu\psi,$$

where D^X is the Dirac operator on Σ_X and D^Y is the Dirac operator on Σ_Y if n is even and the sum of the Dirac operator and its negative if n is odd, and H is the mean curvature of Y in X. A straightforward calculation shows that the Dirac operator D^Y anticommutes with the Clifford multiplication with a normal unit vectorfield ν.

Now let $M_0 = I \times_f S^n$, with $I = [0, L]$ and $f : I \to \mathbb{R}^+$. We denote with Σ_{M_0} the spinor bundle over M_0 and with Σ the spinor bundle over S^n. Then

$$\Sigma_{M_0} \cong \pi^*\Sigma \oplus \pi^*\hat{\Sigma} \quad \text{for } n \text{ odd}, \tag{5}$$

$$\Sigma_{M_0} \cong \pi^*\Sigma \quad \text{for } n \text{ even}, \tag{6}$$

with $\pi : I \times_f S^n \to S^n$.

Let $\{\sigma_j \in \Gamma\Sigma\}_{j\in\mathbb{Z}\setminus\{0\}}$ be an L^2-orthonormal basis of eigenspinors in Σ with $D\sigma_j = k_j\sigma_j$. We choose the numbering of σ_j such that $k_j > 0$ for $j > 0$ and $k_{-j} = -k_j$.

Because of $D^N(\nu \cdot \psi) = -\nu \cdot D^N\psi$ we choose for n even $\sigma_{-j} := \nu \cdot \sigma_j$. For n odd let $\hat{\sigma}_{-j} := \nu \cdot \sigma_j$ be a basis of eigenspinors in $\hat{\Sigma}$. Now we denote with σ_j and $\hat{\sigma}_j$ also the sections in Σ_{M_0} given by the isomorphism (5) and (6).

Then a smooth spinor $\psi \in \Gamma\Sigma_{M_0}$ can be expressed as

$$\psi = \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\theta_j\sigma_j + \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\hat{\theta}_j\hat{\sigma}_j \quad \text{for } n \text{ odd}, \tag{7}$$

$$\psi = \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\theta_j\sigma_j \quad \text{for } n \text{ even}, \tag{8}$$

with $\theta_j, \hat{\theta}_j \in C^\infty([0, L], \mathbb{C})$.

Lemma 2 ([7], p. 919). *A spinor* $\psi = \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\theta_j\sigma_j \in \Gamma\Sigma_{M_0}$, *respectively,* $\psi = \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\theta_j\sigma_j + \Sigma_{j\in\mathbb{Z}\setminus\{0\}}\hat{\theta}_j\hat{\sigma}_j \in \Gamma\Sigma_{M_0}$ *satisfies* $D\psi = \lambda\psi$ *iff* θ_j *and* $\hat{\theta}_j$ *satisfy the equations*

$$\dot{\theta}_j = \left[-\frac{n}{2}\frac{\dot{f}}{f} - \frac{k_j}{f} \right] \theta_j + \lambda\theta_{-j}, \tag{9}$$

$$\dot{\theta}_{-j} = \left[-\frac{n}{2}\frac{\dot{f}}{f} + \frac{k_j}{f} \right] \theta_{-j} - \lambda\theta_j, \tag{10}$$

for all $j \in \mathbb{N}$ for n even and the same equations with θ_{-j} replaced by $\hat{\theta}_{-j}$ for all $j \in \mathbb{Z}\backslash\{0\}$ for n odd.

3. Lower bounds for the eigenvalues and multiplicities

To study the eigenvalues of the Dirac operator on an $SO(n+1)$-manifold $M \cong S^{n+1}$ with $M_0 = I \times_f S^n$ it is sufficient to restrict to analytic functions f because a small deformation of the metric in the C^1-topology does not change the eigenvalues of the Dirac operator too much, more exactly:

Lemma 3 ([7], Proposition 7.1). *For $\epsilon > 0$, $\Lambda > 0$ there exists a C^1-neighbourhood of g, such that for any $\tilde{g}$ in this neighbourhood with associated Dirac operator $\tilde{D}$ and any $\lambda \in [-\Lambda, \Lambda]$ we have*

$$\dim E_{\{\lambda\}}(D) \leq \dim E_{[\lambda-\epsilon,\lambda+\epsilon]}(\tilde{D}) \leq \dim E_{[\lambda-2\epsilon,\lambda+2\epsilon]}(D),$$

where $E_{[a,b]}$ is the direct sum of eigenspaces of D for the eigenvalues $\lambda \in [a, b]$.

Eqs. (9) and (10) have been studied in [18]. The restriction of a smooth spinor on M to $M_0 = I \times_f S^n$ gives a spinor whose coefficient functions θ_j and $\hat{\theta}_j$ in the decomposition (7) and (8) are bounded. The following Lemma generalizes the result of [18].

Lemma 4. *If θ_j and θ_{-j} are solutions of (9) and (10), $k_j > 0$ and θ_j and θ_{-j} are bounded at 0 and L, then $|\theta_{-j}(0)/\theta_j(0)| = \infty$ and $|\theta_{-j}(L)/\theta_j(L)| = 0$.*

Proof. Solutions θ_j and θ_{-j} of (9) and (10) are solutions of

$$\ddot{\theta}_j = -n\frac{\dot{f}}{f}\dot{\theta}_j + \left(\frac{(2k_j + \dot{f})^2 - (n-1)^2\dot{f}^2}{4f^2} - \frac{n}{2}\frac{\ddot{f}}{f} - \lambda^2 \right)\theta_j,$$

$$\ddot{\theta}_{-j} = -n\frac{\dot{f}}{f}\dot{\theta}_{-j} + \left(\frac{(2k_j - \dot{f})^2 - (n-1)^2\dot{f}^2}{4f^2} - \frac{n}{2}\frac{\ddot{f}}{f} - \lambda^2 \right)\theta_{-j}$$

which are regular singular at 0 and L with characteristic indices at 0:

$$\nu_{j,1} = k_j - \frac{n}{2} + 1, \qquad \nu_{j,2} = -k_j - \frac{n}{2},$$

$$\nu_{-j,1} = k_j - \frac{n}{2}, \qquad \nu_{-j,2} = -k_j - \frac{n}{2} + 1,$$

and at L:

$$\nu_{j,1} = k_j - \frac{n}{2}, \qquad \nu_{j,2} = -k_j - \frac{n}{2} + 1,$$
$$\nu_{-j,1} = k_j - \frac{n}{2} + 1, \qquad \nu_{-j,2} = -k_j - \frac{n}{2}. \qquad \square$$

If θ_j and θ_{-j} are solutions of (9) and (10) which are bounded at 0 and L, then $-\theta_{-j}/\theta_j$ is a (possibly singular) solution of the real Riccati equation

$$\dot{z} = \lambda z^2 + \frac{2k_j}{f} z + \lambda \tag{11}$$

with boundary conditions

$$|z(0)| = \infty, \qquad z(L) = 0 \tag{12}$$

Lemma 5 ([18], Lemma 3). *If (11) has a solution with boundary conditions (12) for $k_j > 0$ and $\lambda > 0$, then*

$$\lambda \geq \frac{k_j}{f_{\max}} \tag{13}$$

with $f_{\max} := \max_{t \in [0,L]} f(t)$.

This yields for the Dirac operator:

Theorem 3. *Let $M^{n+1} \cong S^{n+1}$ be a manifold with an effective isometric $SO(n+1)$-action. Let $M_0 = I \times_f S^n$ be the space of principal orbits. Then the spectrum of M is symmetric. The multiplicity of an eigenvalue λ with*

$$\frac{(n/2)+k}{f_{\max}} \leq \lambda < \frac{(n/2)+k+1}{f_{\max}}, \qquad (k \geq 0)$$

is at most $2^{[\frac{n+1}{2}]}\binom{n+k}{k}$.

Especially the smallest positive eigenvalue satisfies $\lambda \geq n/(2f_{\max})$ and has multiplicity at most $2^{[(n+1)/2]}$.

Proof. If $\theta_j\sigma_j + \theta_{-j}\sigma_{-j}$ for n even, respectively $\theta_j\sigma_j + \hat{\theta}_{-j}\hat{\sigma}_{-j}$ for n odd, is an eigenspinor for the eigenvalue λ, then $\theta_j\sigma_j - \theta_{-j}\sigma_{-j}$, respectively $\theta_j\sigma_j - \hat{\theta}_{-j}\hat{\sigma}_{-j}$, is an eigenspinor for the eigenvalue $-\lambda$.

For $k_j = (n/2) + j$ according to (13) the system of equations (9) and (10) has solutions only for $\lambda \geq ((n/2) + j)/f_{\max}$.

For analytic f the space of bounded solutions of (9) and (10) is at most one-dimensional, because the system is weakly singular at 0 and L with characteristic indices $k_j - (n/2)$ and $-k_j - (n/2)$ at 0. Further on the multiplicity of an eigenvalue $k_j = (n/2) + j$ over S^n is $2^{[(n/2)]}\binom{n+j-1}{j}$.

Therefore for n even the multiplicity of an eigenvalue

$$\frac{(n/2)+k}{f_{\max}} \leq \lambda < \frac{(n/2)+k+1}{f_{\max}}$$

is at most

$$\Sigma_{l=0}^{k} 2^{\frac{n}{2}} \binom{n+l-1}{l} = 2^{\frac{n}{2}} \binom{n+k}{k}.$$

For n odd in Eqs. (9) and (10) also $j < 0$ occur. If $\theta_j \sigma_j + \hat{\theta}_{-j} \hat{\sigma}_{-j}$ is an eigenspinor with eigenvalue λ than also $\hat{\theta}_{-j} \sigma_{-j} - \theta_j \hat{\sigma}_j$ because $(\hat{\theta}_{-j}, -\theta_j)$ solves the system (9) and (10) with $k_{-j} = -k_j$. Therefore the multiplicity is at most $2^{\frac{n+1}{2}} \binom{n+k}{k}$. □.

If M is an $(n+1)$-dimensional sphere, which means $M_0 = I \times_{\sin} S^n$, then the well-known multiplicities of the eigenvalues are exactly the maximal multiplicities according to the theorem.

Finally we show that the lower bound of Theorem 2 coincides in the limit with the upper bound (4):

Consider the cylinder $[0, L] \times S^n$ with hemispheres glued to it at the ends. In this example the estimate (2) gives no lower bound. According to (4) there are at least $2^{[(n+1)/2]}$ eigenspinors for the eigenvalue

$$\lambda^2(L) \leq \frac{n^2 L \operatorname{vol}(S^n) + (n+1)^2 \operatorname{vol}(S^{n+1})}{4(L \operatorname{vol}(S^n) + \operatorname{vol}(S^{n+1}))}.$$

For $L \to \infty$ this gives

$$\overline{\lim}_{L\to\infty} \lambda(L) \leq \frac{n}{2}.$$

On the other hand the new lower bound yields

$$\lambda(L) \geq \frac{n}{2} \quad \text{for every } L$$

with multiplicity at most $2^{[(n+1)/2]}$.

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